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AXISYMMETRIC CONTACT PROBLEM IN THE PRESENCE OF A WEDGE*

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The integral equation of the axisymmetric contact problem, obtained in a new form, is used to investigate the case when thin smooth rigid axisymmetric wedges (posts) are inserted into the bodies making contact along their common axis of symmetry. Singularities are established in the solution, particularly the possibility of the origination of negative drawing together of the bodies and the annular pressure domain under the action of a flat stamp of circular planform on the half-space.

1. Action of a wedge and an overload on a half-space. Let a thin smooth circular wedge of radius $h(z)$ be introduced on a section L along the z-axis on a half-space $z \geqslant 0$, and let a normal load $p(\rho)$ act on its boundary $z=0$. Determine the state of stress and strain of the half-space. It is assumed that the required elastic displacements of its points are constrained at infinity.

The complex potentials corresponding to the problem formulated can be found if potentials for the corresponding plane problem /1,2/ and the relation between the solutions of the plane and axisymmetric problems /3/ are used. We obtain

$$\Phi'(\Omega)=2q_0\pi^{-1}\int_L \eta h'(\eta)(\eta^2-\Omega^2)^{-1}d\eta \tag{1.1}$$

$$\Psi'(\Omega)=2q_0\pi^{-1}\Big[-\int_L \eta h'(\eta)(\eta+\Omega)^{-2}d\eta+\int_0^\infty p^*(\eta)(\eta^2+\Omega^2)^{-1}d\eta\Big]$$

$$\Omega=z+i\xi,\ \ \xi=\rho\cos\theta,\ \ q_0=2\mu(1+k_0)^{-1},\ \ k_0=\mu(\lambda+\mu)^{-1}$$

Here $p^*(\eta)$ is the transform of $p(\rho)$ in the sense of /3/.
The solution is written in displacements in the form

$$\begin{Bmatrix}u\\ v\end{Bmatrix}=\begin{Bmatrix}x\\ y\end{Bmatrix}\frac{1}{\rho}\langle\alpha u_0(\rho\alpha,z)\rangle,\qquad w=\langle w_0(\rho\alpha,z)\rangle \tag{1.2}$$

$$\begin{Bmatrix}u_0\\ w_0\end{Bmatrix}=\operatorname{Re}\,[k^{\mp}\Phi\mp i\xi\Phi'+k^{\pm}\Psi\pm z\Psi']\begin{Bmatrix}-i\\ 1\end{Bmatrix} \tag{1.3}$$

$$\rho^2=x^2+y^2,\ \ \alpha=\cos\theta,\ \ k^+=k_0,\ \ k^-=1+k_0$$

The angular brackets denote integration with respect to θ between 0 and 2π. Correspondingly, we deduce /2/

$$\begin{Bmatrix}\sigma_\rho\\ \sigma_\theta\end{Bmatrix}=\Big\langle\sigma_\xi^\circ-2\mu\alpha_\pm^2\frac{\partial u^\circ}{\partial\xi}\Big\rangle\quad\begin{pmatrix}\alpha_+=\sin\theta\\ \alpha_-=\alpha\end{pmatrix} \tag{1.4}$$

$$\sigma_z=\langle\sigma_z^\circ\rangle,\ \ \tau_{z\rho}=\langle\alpha\tau_{z\xi}^\circ\rangle$$

$$\begin{Bmatrix}\sigma_\xi^\circ\\ \sigma_z^\circ\end{Bmatrix}=\operatorname{Re}\,[\Phi'\mp i\xi\Phi''+\Psi'\pm z\Psi''],\qquad \overset{\circ}{\tau}_{z\xi}=-\operatorname{Re}\,[\xi\Phi''+iz\Psi'']$$

The remaining stress components are also easily found.

It is seen from (1.1)– (1.4) that the solution of the problem under consideration is obtained by superposition of solutions corresponding to the action of just the wedge and just the load.

We shall later need values of Φ,Ψ corresponding to the action of just a normal lumped load, distributed uniformly along a circle of radius R and having a result equal to P, on the half-space.

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Since $h(z) = 0$, we deduce at once: $\Phi'(\Omega) = 0$. The values of Ψ' are related to evaluation of the integral in which

$$p^*(\eta) = \begin{cases} 0, & |\eta| < R \\ (2\pi)^{-1}[(\eta^2 - R^2)^{-1/2}]_\eta', & |\eta| > R \end{cases}$$

However, it is simpler to find the function directly by starting from the boundary condition

$$\mathrm{Re}\Psi'(t) = p^*(\xi), \quad t = i\xi$$

It can be rewritten in the form

$$\mathrm{Re}\Psi'(t) = -(2\pi)^{-2}P\,[(t^2 + R^2)^{-1/2}]_t'$$

Here the branch taking on the value t for large t is understood under the radical. Considering the required function $\Psi(\Omega)$ to vanish at infinity, we deduce as a result of analytic continuation

$$\Psi(\Omega) = -(2\pi)^{-2}P\,(\Omega^2 + R^2)^{-1/2}$$

Substituting the values found for the complex potentials into (1.3) and then into (1.2), we obtain

$$w(\rho, 0) = \begin{cases} P(2\pi)^{-2}q_0^{-1}\langle (R^2 - \rho^2\cos^2\theta)^{-1/2}\rangle, & \rho < R \\ P(2\pi)^{-2}q_0^{-1}\langle (\rho^2 - R^2\cos^2\theta)^{-1/2}\rangle, & \rho > R \end{cases} \tag{1.5}$$

$$(w(\rho, 0) \sim \ln|\rho - R|, \quad \rho \to R)$$

The elastic displacements of points of the half-space boundary in the directions of the x and y axes corresponding to the load under consideration of particular form, are zero.

2. Integral equation of the axisymmetric contact problem for $h(z) = 0$. It is known that the problem in the Hertz formulation reduces to seeking the axisymmetric normal pressure $p(\rho)$ in a circle, ring (or in their combination) under the condition that the elastic displacements normal to the boundary are given in the domain mentioned.

Let a circle of radius ρ_1 lie entirely in the pressure domain. Then we have a uniformly distributed lumped load with resultant $2\pi p(\rho_1)\rho_1 d\rho_1$ at points of this circle and a corresponding displacement of half-space points normal to the boundary dw. Integrating with respect to ρ_1 between the limits 0 and b (when the pressure domain is a circle of radius b), we derive an integral equation of the axisymmetric contact problem under consideration by using the relationship (1.5)

$$\left\langle \int_0^\rho p(\rho_1)(\rho^2 - \rho_1^2\cos^2\theta)^{-1/2}\rho_1\,d\rho_1 + \int_\rho^b p(\rho_1)(\rho_1^2 - \rho^2\cos^2\theta)^{-1/2}\rho_1\,d\rho_1 \right\rangle = f(\rho) \tag{2.1}$$

$$f(\rho) = 2\pi q_0 w(\rho, 0)$$

Its solution is known and is determined under the conditions that $f(\rho)$ and $f'(\rho)$ are continuous in the interval $0 < \rho \leqslant b$, by the formula /4/

$$p(\rho) = (2\pi)^{-1}\left[F(b)(b^2 - \rho^2)^{-1/2} - \int_\rho^b F'(s)(s^2 - \rho^2)^{-1/2}\,ds\right] \tag{2.2}$$

$$F(\rho) = 2\pi^{-1}\left[f(0) + \rho\int_0^\rho f'(\sigma)(\rho^2 - \sigma^2)^{-1/2}\,d\sigma\right]$$

Let us show that the relationship (2.2) turns (2.1) into an identity. Indeed, after substitution, interchange of the order of integration, and evaluation of the inner integrals, we obtain

$$(2\pi)^{-1}\left\langle F(b)\,\chi(\rho; b; \theta) + \int_0^\rho F'(s)\,T(\rho; s; \theta)\,ds + \int_\rho^b F'(s)\,\chi(\rho; s; \theta)\,ds \right\rangle = f(\rho) \tag{2.3}$$

$$\chi(\rho;\eta;\theta) = (2\cos\theta)^{-1}\ln|\rho\sin\theta - (\eta^2-\rho^2)^{1/2}\|\rho\sin\theta + (\eta^2-\rho^2)^{1/2}| +$$
$$\pi/2 - \operatorname{arctg}[\rho(\eta^2-\rho^2)^{-1/2}\sin\theta] - T(\rho;\eta;\theta)$$
$$T(\rho;\eta;\theta) = (2\cos\theta)^{-1}\ln|\rho - \eta\cos\theta\|\rho + \eta\cos\theta|^{-1}$$

Integration by parts reduces equation (2.3) to the form

$$\int_0^\rho F(s)(\rho^2 - s^2)^{-1/2}ds = f(\rho)$$

which proves the assertion /4/.

3. Axisymmetric contact problem in the presence of a wedge. Let two bodies of revolution, initially tangent at one point O, be subjected to compressive forces P directed along the common normal at the point O to surfaces bounding the bodies and axisymmetric thin smooth wedges (slits) directed along the same normal Fig.1. It is assumed that the slits are strictly within the bodies, and the dimensions of the pressure area are small compared to the body size so that the latter can be replaced by half-spaces.

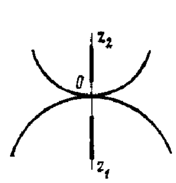

Fig.1

At the point O directing the $z_j\ (j = 1, 2)$ axes along the mentioned normal into the bodies making contact, we obtain the total normal elastic displacements of the boundary points of the bodies because of their deformation

$$w_j = w_j^{(1)} + w_j^{(2)}\ (j = 1, 2) \tag{3.1}$$

Here $w_j^{(1)}$ are the elastic displacements due to body compression, and $w_j^{(2)}$ are the displacements because of the action of the wedges.

Reasoning in the usual manner, we deduce the fundamental relationship /4/

$$w_1 + w_2 = \alpha - z_1^\circ(\rho) - z_2^\circ(\rho) \tag{3.2}$$

Here α is the approach of the bodies, while $z_j^\circ(\rho)$ are known and determined by giving the body shapes prior to deformation.

After replacement of the bodies by half-spaces, the problem reduced to seeking the total pressure $p(\rho)$ in a circle, ring, or their combination. The displacements $w_j^{(1)}$ hence become known.

For given wedge shapes and locations on the axes, we obtain /2/

$$w_j^{(2)}(\rho, 0) = \int_{L_j} h_j'(\eta)(\rho^2 + \eta^2)^{-1/2}\eta\, d\eta$$

The displacements governed by the pressure are written, for instance, in the case when the pressure domain is the circle $0 \leqslant \rho \leqslant b$, are written in the form (2.1). Therefore substituting into (3.2), we derive an integral equation of the axisymmetric contact problem in the presence of wedges

$$\left\langle \int_0^\rho p(\rho_1)(\rho^2 - \rho_1^2\cos^2\theta)^{-1/2}\rho_1\, d\rho_1 + \int_\rho^b p(\rho_1)(\rho_1^2 - \rho^2\cos^2\theta)^{-1/2}\rho_1\, d\rho_1 \right\rangle = f_1(\rho)$$

$$f_1(\rho) = 2\pi(q_{10} + q_{20})\left[\alpha - \sum_{j=1}^2\left\{z_j^\circ + \int_{L_j} h_j'(\eta)(\eta^2 + \rho^2)^{-1/2}\eta\, d\eta\right\}\right]$$

If $f_1(\rho)$ and $f_1'(\rho)$ are continuous in the interval $0 < \rho \leqslant b$, then this solution is written in the form (2.2).

For a given b the approach α is found from the condition of equivalence of the pressure to the compressing force P. In the case of boundedness of the solution for $\rho = b$, we have the additional condition $F(b) = 0$ governing the magnitude of b.

4. Action of a stamp and a wedge on a half-space. As an illustration, we examine the problem of the combined action on a half-space, of a stamp of circular planform and a circular wedge of constant section of radius h that occupies a segment $H_1 \leqslant z \leqslant H_2,\ H_1 > 0$ along the z axis.

This latter can be obtained from a wedge with conical ends by a passage to the limit.
On the basis of /2,4/ as well as (3.3) we have

$$z_j^\circ = 0\ (j = 1, 2),\ q_{20} = 0,\ q_{10} = q_0,\ w_2 = 0 \tag{4.1}$$
$$w_1^{(2)}(\rho, 0) = H_1(\rho^2 + H_1^2)^{-1/2} - H_2(\rho^2 + H_2^2)^{-1/2}$$
$$f_1(\rho) = 2\pi q_0[\alpha + h\{H_2(\rho^2 + H_2^2)^{-1/2} - H_1(\rho^2 + H_1^2)^{-1/2}\}]$$

Since $f_1(\rho), f_1'(\rho)$ are continuous in the interval $0 \leqslant \rho \leqslant b$, then the solution of (3.4) under the condition that the required pressure is non-negative in the interval mentioned is written in the form (2.2). We find

$$F(\rho) = 4q_0\{\alpha + h[H_2^2(\rho^2 + H_2^2)^{-1} - H_1^2(\rho^2 + H_1^2)^{-1}]\} \tag{4.2}$$

In recent years, the dimensionless quantities $\pi(2q_0)^{-1}p(\rho)$, α/b, h/b, h/H_j, ρ/H_j, b/H_j, ρ/b are understood to be, respectively, the quantities $p(\rho), \alpha, h, h_j, \rho_j, b_j, \rho$. In these notations the required dimensionless pressure calculated by means of formula (2.2) is written in the form

$$p(\rho) = [\alpha + h(a_2 - a_1)](1 - \rho^2)^{-1/2} + \chi_2(\rho) - \chi_1(\rho) \tag{4.3}$$
$$\chi_j(\rho) = h_j(1 + \rho_j^2)^{-1/2}\{\operatorname{arctg}[(b_j^2 - \rho^2)^{1/2}(1 + \rho_j^2)^{-1/2}] +$$
$$a_j(1 + \rho_j^2)^{1/2}(b_j^2 - \rho_j^2)^{1/2}\},\ a_j = (1 + b_j^2)^{-1}$$

For $h = 0$ we obtain the known solution /4/.

Let us indicate the sufficient condition of positiveness of the pressure $p(\rho)$ determined by formula (4.3) in the interval $0 \leqslant \rho \leqslant b$. We consider the function

$$\Phi_1(\rho) = [\alpha + h(a_2 - a_1)](1 - \rho^2)^{-1/2} - \chi_1(\rho) \tag{4.4}$$

It can be established that $\Phi_1'(\rho) > 0$, i.e., $\Phi_1(\rho)$ is a monotonically increasing function in the interval mentioned. Hence, $\Phi_1(\rho) \geqslant 0$ if $\Phi_1(0) \geqslant 0$. Under this condition we have

$$p(\rho) = \Phi_1(\rho) + \chi_2(\rho) \geqslant 0,\ \chi_2(\rho) \geqslant 0$$

Thus, for the non-negativity of $p(\rho)$ in the interval $0 \leqslant \rho < 1$ it is sufficient to satisfy the inequality

$$\Phi_1(0) = \alpha - h(\psi_1 + a_2b_2) \geqslant 0 \tag{4.5}$$
$$\psi(x) = x \operatorname{arctg} x,\ \psi_j = \psi(b_j)\ (j = 1, 2)$$

From (4.5) there follows that $\alpha \geqslant 0$ for any $H_j, b, H_1 > 0$.

Let P be the compressing force on the stamp. Denoting the dimensionless compressing force $\pi P/(2q_0, b^2)$ by the same letter, we write a relationship governing the approach between the bodies α by using (4.3) α ($\varphi(x)$ is monotonically decreasing function)

$$P = \alpha + h(\varphi_2 - \varphi_1);\ \varphi(x) = x^{-1}\operatorname{arctg} x,\ \varphi_j = \varphi(b_j) \tag{4.6}$$

Substituting the value of α from (4.6) into condition (4.5) we write it in the form of a constraint imposed on the force compressing the stamp

$$P \geqslant h(\varphi_2 - \varphi_1 + \psi_1 + a_2b_2^2) \tag{4.7}$$

Let us examine the case of a semi-infinite wedge: $H_2 = \infty$, $H_1 > 0$. We obtain

$$b_2 = \chi_2 = \psi_2 = 0,\ a_2 = \varphi_2 = 1$$

Condition (4.5) becomes here necessary also.

The solution (4.3) is written in the form

$$p(\rho) = (\alpha + ha_1b_1^2)(1 - \rho^2)^{-1/2} - \chi_1(\rho) \tag{4.8}$$

if the following inequality holds

$$P \geqslant h(1 - \varphi_1 + \psi_1)$$

If it is violated (the force compressing the stamp is not large enough), a gap is formed between the stamp and the deformed boundary of the half-space in the neighborhood of the point $z = 0$, the area of the pressure acquires the shape of a flat ring with an unknown inner boundary and the pressure distribution under the stamp must be sought be starting from this condition. We do not consider this in this paper.

We note that the solution (4.8) retains meaning even in the limit case when $H_1 \to 0$ under the condition that $\rho \neq 0$. Indeed, in the limit we have

$$b_1 = \infty,\ a_1 = 0,\ a_1 b_1^2 = 1,\ \chi_1 = \varphi_1 = 0$$

The equations (4.6) and (4.8) are written in the form

$$P = \alpha + h,\quad p(\rho) = (\alpha + h)(1 - \rho^2)^{-1/2}$$

We hence deduce an expression for the pressure

$$P(\rho) = P(1 - \rho^2)^{-1/2},\quad \alpha = P - h$$

It is seen that at all points of the circle, with the exception of the center, it agrees with the ordinary pressure. At the center of the circle, i.e., for $\rho = 0$, $H_1 \to 0$, as is seen from (4.8) the pressure becomes unlimited. For $P < h$ the approach of the bodies α turns out to be negative.

We obtain another particular case by letting the solution H_1 in (4.3) approach zero under the condition that $\rho \neq 0$. Here $\chi_1 = a_1 = 0$, and the solution (4.3) takes the form

$$p(\rho) = (\alpha + ha_2)(1 - \rho^2)^{-1/2} + \chi_2(\rho),\quad P = \alpha + h\varphi_2$$

under the condition that $\alpha + ha_2 > 0$, i.e., $\alpha = P - h\varphi_2 \geqslant 0$. It will even be satisfied for negative α if the force P satisfies the condition

$$h(\varphi_2 - a_2) < P < h\varphi_2$$

For $P \leqslant h(\varphi_2 - a_2)$ we should set $\alpha + ha_2 = 0$ and the solution will be written in the form

$$p(\rho) = \chi_2(\rho)$$

The unknown b_2 governing the size of the pressure domain is found from the equation

$$P - h(\varphi_2 - a_2) = 0$$

which has a unique solution for given P.

For $\rho = 0$ and $H_1 \to 0$ the solution (4.3) does not tend to a finite limit.

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